An Introduction to Stochastic Processes

Anton Kliewer

Cooperative Institute for Research in the Atmosphere

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Outline

Random Walk

A patron leaves a bar and takes a step to the left or right with equal probability.

His home his 20 steps to his left.

What is the probability he gets home?

Random Walk

Start at $x = 0$.

In every fixed interval of time Δt, move some random distance Δx_j, where the Δx_j are independent Gaussian random variables with variance v.

In time $T = N\Delta t$ the total distance moved is $W = \sum_{i=1}^{n} \Delta x_j$.

Random Walk

Expectation and Variance of $W = \sum_{i=1}^{n} \Delta x_j$?

$E(W) = 0$. Why?

Random Walk

Expectation and Variance of $W = \sum_{i=1}^{n} \Delta x_j$?

$E(W) = 0$. Why?

$E(W^2) = vN = vT/\Delta t$

Note that terms of the form $E(\Delta x_i \Delta x_j)$ vanish due to the independence assumption. Letting $v = \sigma^2 \Delta t$, in the limit as $\Delta t \rightarrow 0$ we have that $E(W^2) = \sigma^2 T$.

Let Ω denote the sample space of all possible outcomes of a random experiment.

Let $\mathcal{A}$ be a σ-algebra, i.e.

- $\Omega \in \mathcal{A}$
- $E^c \in \mathcal{A}$ if $E \in \mathcal{A}$
- $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$ if $E_1, E_2, \ldots \in \mathcal{A}$

Let the measure $\mathcal{P} : \mathcal{A} \to [0, 1]$ be such that

- $\mathcal{P}(\Omega) = 1$
- $\mathcal{P}(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathcal{P}(E_i)$ if $E_i \bigcap E_j = \emptyset$ for $i \neq j$

We have defined our sample space $(\Omega, \mathcal{A}, \mathcal{P})$

Define the random variable $X : \Omega \to \mathbb{R}$. So for every outcome $\omega \in \Omega$, $X(\omega)$ is a real number.

We assume that X is measurable and that the distribution function is defined as $F_X(x) = \mathcal{P}(\omega \in \Omega : X(\omega) \leq x)$

This defines our probability density function p by

$$\mathcal{P}(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b p(s)ds$$

The expectation is the first moment defined by

$$E(X) = \int_{-\infty}^{\infty} xp(x)dx$$

and is referred to as the mean μ.

The variance is found by computing the second moment

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p(x)dx$$

and is then defined as

$$Var(X) = E((X-\mu)^2) = E(X^2) - \mu^2$$

Stochastic Process

For our given sample space $(\Omega, \mathcal{A}, \mathcal{P})$ a stochastic process is a family of random variables $\{X(t), t \in \tau\}$ where the set τ is continuous.

So we have $X : \tau \times \Omega \longrightarrow \mathbb{R}$ and that $X(t) = X(t, \omega)$ is a random variable for each $t \in \tau$ and $X(t, \omega)$ is a function of t for each $\omega \in \Omega$.

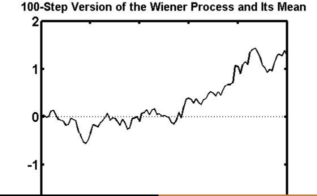
The typical example is a Wiener Process.

Definition

Wiener Process: a continuous stochastic process with stationary independent increments such that

- $W(0) = 0$
- $W(t) - W(s) \sim N(0, t - s)$ for all $0 \leq s \leq t$

It is also called Brownian motion after the botanist Robert Brown. In 1828 while looking at pollen grains under a microscope he noticed tiny particles demonstrating continuous, jittery motion.



History of Brownian Motion

1900: Integrated into a study of the stock market

1905: Albert Einstein estimated Avogadro's number to be 6.1×10^{23}.

1923: Norbert Wiener proved existence and construction

1973: Black-Scholes mathematically described European-style options

In mathematical modeling many times the independent variable is given by t, representing time, and the dependent variable is X. We can rewrite a differential equations of the form

$$\frac{\partial X(t)}{\partial t} = f(t, X(t))$$

as

$$\partial X(t) = f(t, X(t))\,\partial t$$

If we add a stochastic term to the previous equation we may have a more accurate representation of our observered phenonmena. Our equation would then become

$$\partial X(t) = f(t, X(t))\,\partial t + g(t, X(t))\,\partial W(t)$$

$$\partial X(t)) = f(t, X(t))\,\partial t + g(t, X(t))\,\partial W(t)$$

- f is called the drift coefficient (function) and g is called the diffusion coefficient (function).
- ∂W is the Wiener process.
- Each simulation or realization will be different.

So immediately we have the following questions:

- How do we interpret and construct dW(t)?
- Where does g come from?
- What is a solution? How do we find it?
- What problems should we use stochastic differential equations (SDE's) for?

Itô's Lemma

Now lets consider the following stochastic differential equation

$$\partial X(t,\omega) = f(t, X(t,\omega))\partial t + g(t, X(t,\omega))\partial W(t,\omega)$$

In this equation the function

- ω is a trajectory
- ∂W is the Wiener process

Itô's lemma states that if $F(t, X(t))$ has continuous derivatives $\frac{\partial F(t,x)}{\partial t}, \frac{\partial F(t,x)}{\partial x}, \frac{\partial^2 F(t,x)}{\partial x^2}$ then

$$\partial F(t, X(t)) = \left[\frac{\partial F(t,X)}{\partial t} + f(t,X)\frac{\partial F(t,X)}{\partial x} + \frac{1}{2}g^2(t,X)\frac{\partial^2 F(t,X)}{\partial x^2}\right]\partial t + g(t,X)\frac{\partial F(t,X)}{\partial x}\partial W(t)$$

What does Itô's lemma do?

Due to the Brownian motion there is not a unique solution. Our goal is to then find the probability distribution of all solutions.

Itô's lemma allows us to quickly calculate the moments of this distribution.

Alternatively we could solve the Fokker-Planck differential equation for the corresponding stochastic differential equation.

Example

Consider

$$\partial X(t) = \partial t + X(t)\partial W(t) \quad X(0) = 0$$

By integrating we get

$$X(t) = t + \int_0^t X(s)\partial W(s)$$

By finding the expected value of both sides we get

$$E(X(t)) = t$$

If we consider $F(t, X) = X^2$ and apply Itô's formula we get

$$\partial(X^2(t)) = \left[2X(t) + X^2(t)\right]\partial t + 2X^2(t)\partial W(t)$$

so that

$$E(X^2(t)) = E\int_0^t (2X(s) + X^2(s))\partial s$$

Example

We can really think of

$$E(X^2(t)) = E\int_0^t (2X(s) + X^2(s))\partial s$$

as the differential equation

$$\begin{aligned}\frac{\partial E(X^2(t))}{\partial t} &= 2E(X(t)) + E(X^2(t)) \\ &= 2t + E(X^2(t))\end{aligned}$$

with $E(X^2(0)) = 0$. Solving this gives the exact second moment for X(t)

$$E(X^2(t)) = -2t - 2 + 2e^t$$

Example

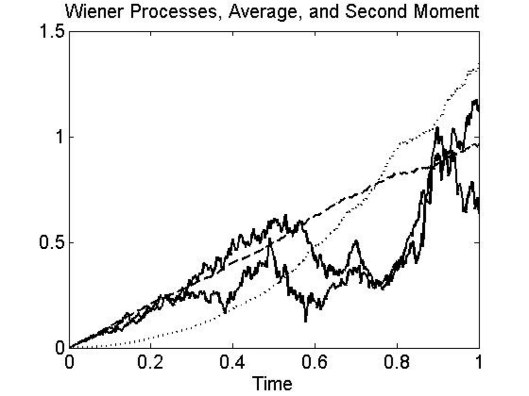
So here we know the mean to be

$$E(X(t)) = t$$

and we can calculate the variance from the second moment to be

$$\begin{aligned} Var(X(t)) &= E(X^2(t)) - (E(X(t)))^2 \\ &= 2e^t - 2 - 2t - t^2 \end{aligned}$$

	Exact Results (t=1)	Simulated Results (t=1)
Mean	1.0	0.9631
Variance	0.4365	0.4215



If the exact solution of a stochastic differential equation cannot be determined, which is often the case, numerical methods exist to approximate the solution. The first one we'll consider is the Euler-Maruyama method which is given by

$$X_{i+1} = X_i(\omega) + f(t_i, X_i(\omega))\Delta t + g(t_i, X_i(\omega))\Delta W_i(\omega)$$
$$X_0(\omega) = X(0, \omega)$$

for $i = 0, 1, 2, \ldots, N-1$
where $X_i(\omega) \approx X(t_i, \omega)$
$t_i = i\Delta t$
$\Delta t = T/N$
$\Delta W_i(\omega) = (W(t_{i+1}, \omega) - W(t_i, \omega)) \sim N(0, \Delta t)$ The mean square error is proportional to Δt

Milstein's method is a second order method with mean square error proportional to $(\Delta t)^2$.

$$X_{i+1}(\omega) = X_i(\omega) + f(t_i, X_i(\omega))\Delta t + g(t_i, X_i(\omega))\Delta W_i(\omega) + \frac{1}{2} g(t_i, X_i(\omega)) \frac{\partial g(t_i, X_i(\omega))}{\partial x} [(\Delta W_i(\omega))^2 - \Delta t]$$

$X_0(\omega) = X(0, \omega)$

for $i = 0, 1, 2, \ldots, N-1$
with $X_i(\omega) \approx X(t_i, \omega)$
$t_i = i\Delta t$
$\Delta t = T/N$
$\Delta W_i(\omega) = (W(t_{i+1}, \omega) - W(t_i, \omega)) \sim N(0, \Delta t)$

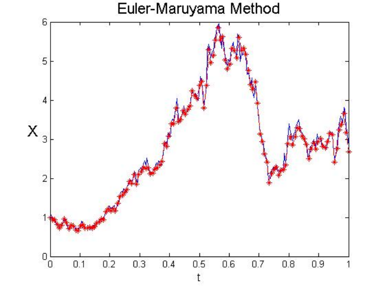
Consider the SDE given by

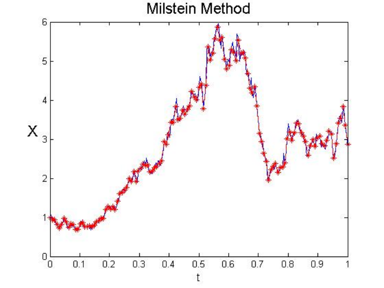
$$\partial X = \lambda X \partial t + \mu X \partial W, \qquad X(0) = X_0$$

An exact solution is known and is given by

$$X(t) = X(0) \exp[(\lambda - \tfrac{1}{2}\mu^2)t + \mu W(t)]$$

Choosing $\lambda = 2$, $\mu = 1$, and $X_0 = 1$ and applying the Euler-Maruyama and Milstein Methods gives the following results.





Method	Value at $t = 1$
Solution	2.8364
Euler-Maruyama	2.6769
Milstein	2.8655

The error for Euler-Maruyama is 0.1595 and Milstein is 0.0291 at the endpoint.

Once again if we consider the SDE

$$\partial X(t) = f(t, X(t))\partial t + g(t, X(t))\partial dW(t)$$

and apply Itô's formula to $F \in C_0^\infty(\mathbb{R})$ to get

$$\partial F(X) = \left[f(t, X)\frac{\partial F(t, X)}{\partial x} + \frac{1}{2}g^2(t, X)\frac{\partial^2 F(t, X)}{\partial x^2} \right] \partial t + g(t, X)\frac{\partial F(t, X)}{\partial x}\partial W(t)$$

Taking expectations of both sides gives

$$\frac{\partial E(F)}{\partial t} = E\left[\frac{\partial F}{\partial x} f + \frac{1}{2} g^2 \frac{\partial^2 F}{\partial x^2} \right]$$

But recall that expectation can also be found by integrating against the distribution function. Let $\rho(t, x)$ be the density for solutions. Then

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\rho(t,x)F(x)dx = \int_{-\infty}^{\infty}\rho(t,x)\left[\frac{\partial F}{\partial x}f + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2}\right]dx$$

If you integrate the right hand side by parts we get

$$\int_{-\infty}^{\infty}F(x)\left[\frac{\partial\rho(t,x)}{\partial t} + \frac{\partial(\rho(t,x)f(t,x))}{\partial x} - \frac{1}{2}\frac{\partial^2(\rho(t,x)g^2(x,t))}{\partial x^2}\right]dx = 0$$

Since this holds $\forall F \in C_0^{\infty}(\mathbb{R})$ then

$$\frac{\partial\rho(t,x)}{\partial t} = -\frac{\partial(\rho(t,x)f(t,x))}{\partial x} + \frac{1}{2}\frac{\partial^2(\rho(t,x)g^2(t,x))}{\partial^2 x}$$

Example

Consider the stochastic differential equation

$$\partial X(t) = \partial W(t)$$
$$X(0) = x_0$$

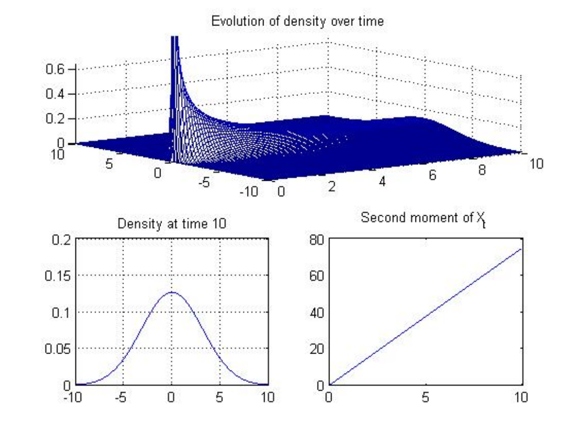
The corresponding Fokker-Planck equation for the density of solutions is given by

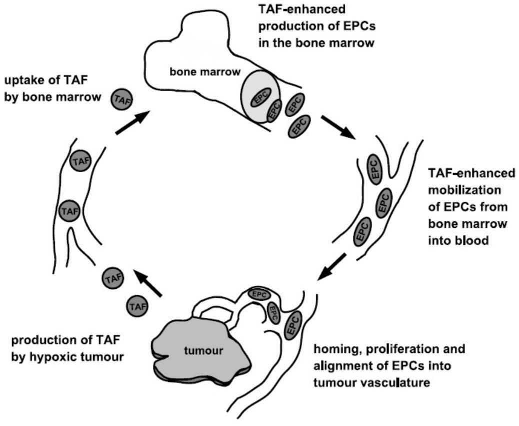
$$\frac{\partial \rho(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2(\rho(t,x))}{\partial x^2}$$
$$\rho(0,x) = \delta(x - x_0)$$

This of course has the solution

$$\rho(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-x_0)^2}{2t}\right]$$

A second order Runge-Kutta method gives the following Matlab results





Tumour-induced vasculogenesis: EPC's are produced in the bone marrow while the tumor secretes TAF. *Stamper I.J., Byrne H.M., Owen M.R., Maini P.K.*

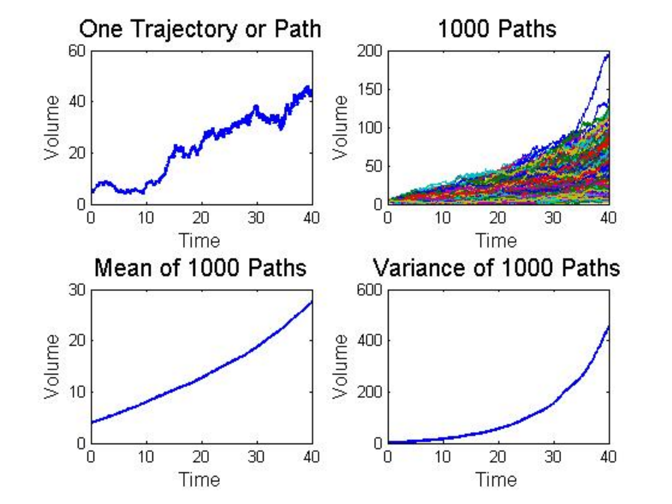
Table: Model Variables

Variable	Dimension	Description
t	T	Time
m	l^3	Tumor volume
u	l^3	Vasculogenesis-derived tumor vasculature
w	l^3	Angiogenesis-derived tumor vasculature
v	l^3	Total volume of tumor vasculature
x	1	Number of EPC's in bone marrow
y	1	Number of EPC's in blood
z	1	Number of EPC's in tumor

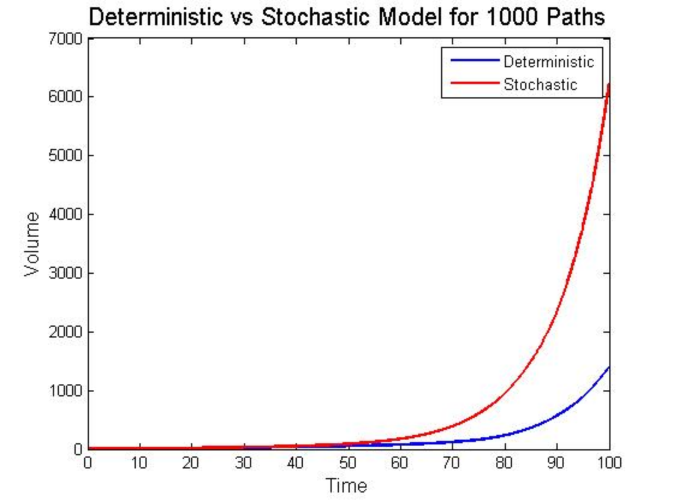
Full Model

$$
\begin{aligned}
\frac{dx}{dt} &= p_1 + p_2 m H(m - \Gamma v) - k_1 x - k_2 m x H(m - \Gamma v) - d_1 x \\
\frac{dy}{dt} &= k_1 x + k_2 m x H(m - \Gamma v) - k_3 (v + r) m y H(m - \Gamma v) - d_2 y \\
\frac{dz}{dt} &= k_3 (v + r) m y H(m - \Gamma v) + \frac{p_3 (v + r) m z H(m - \Gamma v)}{v} - k_4 z - d_3 z \\
\frac{du}{dt} &= k_4 z + \frac{p_5 (v + r) m u H(m - \Gamma v)}{v} - \frac{\delta_3 m u}{m + \delta_4 v} - d_5 u \\
\frac{dw}{dt} &= \frac{p_4 (v + r) m (1 + w) H(m - \Gamma v)}{v} - \frac{\delta_1 m w}{m + \delta_2 v} - d_4 w \\
\frac{dm}{dt} &= \left(\frac{v}{v + m} - d_0 \right) m \\
v &= w + u + 1
\end{aligned}
$$

Full Stochastic Model



Full Model

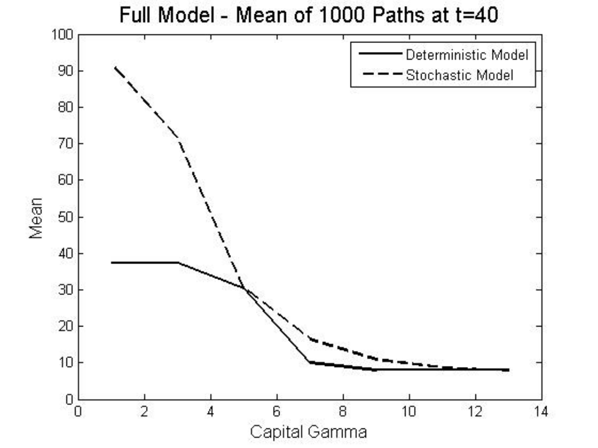

Deterministic vs Stochastic Model for 1000 Paths
Deterministic
Stochastic
Volume
Time
7000
6000
5000
4000
3000
2000
1000
0
10
20
30
40
50
60
70
80
90
100

The growth of a tumor depends on the blood flow it receives from the body. Vascularization is the formation of new blood vessels. The two ways new blood vessels can form are:

Angiogenesis - formation of new blood vessels from pre-existing ones

Vasculogenesis - formation of new blood vessels when there are no pre-existing ones

Vasculogenesis plays a much larger role in the stochastic model than in the deterministic model.

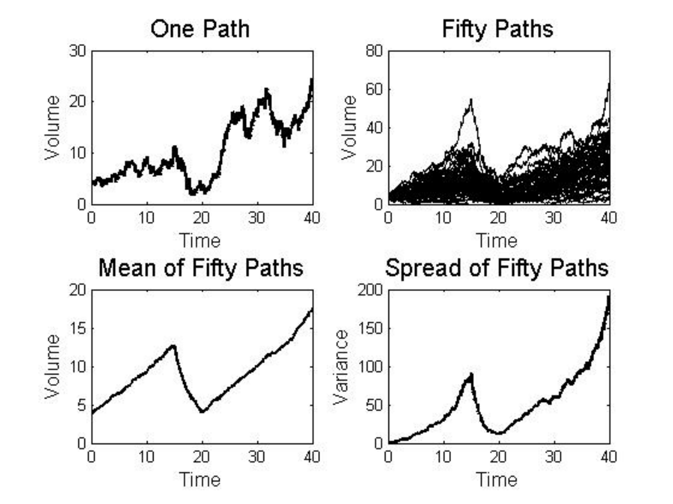


Γ represents the value of m/v above which TAF production is active

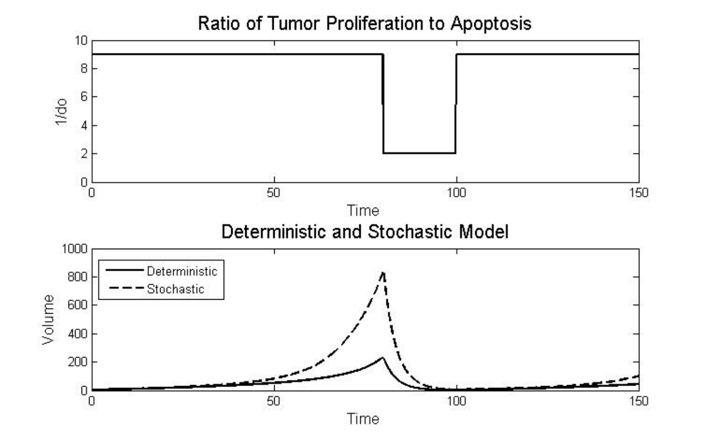
The question then becomes what forms of treatment can we apply to a patient in order to eradicate the tumor or maintain it in an avascular state.

Recall that the parameter d_0 is the per capita death rate, or apoptosis, of tumor volume. If we increase this parameter for a period of time representing a chemotherapy treatment what will happen to the tumor volume?

Effect of a Chemotherapy Treatment



Effect of a Chemotherapy Treatment



Stochastic differential equations combine aspects from probability and statistics, analysis, and differential equations.

The real world includes a multitude of variables that generally go unobserved in basic models.

If the stochastic model's results are different than the deterministic model then interesting conclusions may be found.

SDE's can answer a multitude of questions and be applied to numerous fields.

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Questions?